THE CHROMATIC NUMBER OF ALMOST STABLE KNESER HYPERGRAPHS

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Abstract. Let $V(n,k,s)$ be the set of k-subsets S of $[n]$ such that for all $i,j\in S$, we have $|i-j|\geq s$ We define almost s-stable Kneser hypergraph $KG^r\binom{[n]}{k}^{\sim}_{s\text{-stab}}$ to be the r-uniform hypergraph whose vertex set is $V(n,k,s)$ and whose edges are the r-uples of disjoint elements of $V(n,k,s)$.

With the help of a Z_p-Tucker lemma, we prove that, for p prime and for any $n\geq kp$, the chromatic number of almost 2-stable Kneser hypergraphs $KG^p\binom{[n]}{k}^{\sim}_{2\text{-stab}}$ is equal to the chromatic number of the usual Kneser hypergraphs $KG^p\binom{[n]}{k}$, namely that it is equal to $\left\lceil\frac{n-(k-1)p}{p-1}\right\rceil$.

Defining $\mu(r)$ to be the number of prime divisors of r, counted with multiplicities, this result implies that the chromatic number of almost $2^{\mu(r)}$-stable Kneser hypergraphs $KG^r\binom{[n]}{k}^{\sim}_{2^{\mu(r)}\text{-stab}}$ is equal to the chromatic number of the usual Kneser hypergraphs $KG^r\binom{[n]}{k}$ for any $n\geq kr$, namely that it is equal to $\left\lceil\frac{n-(k-1)r}{r-1}\right\rceil$.

1. Introduction and main results

Let $[a]$ denote the set $\{1,\ldots,a\}$. The Kneser graph $KG^2\binom{[n]}{k}$ for integers $n\geq 2k$ is defined as follows: its vertex set is the set of k-subsets of $[n]$ and two vertices are connected by an edge if they have an empty intersection.

Kneser conjectured [6] in 1955 that its chromatic number $\chi\left(KG^2\binom{[n]}{k}\right)$ is equal to $n-2k+2$. It was proved to be true by Lovász in 1979 in a famous paper [7], which is the first and one of the most spectacular application of algebraic topology in combinatorics.

Soon after this result, Schrijver [11] proved that the chromatic number remains the same when we consider the subgraph $KG^2\binom{[n]}{k}_{2\text{-stab}}$ of $KG^2\binom{[n]}{k}$ obtained by restricting the vertex set to the k-subsets that are 2-*stable*, that is, that do not contain two consecutive elements of $[n]$ (where 1 and n are considered to be also consecutive).

Let us recall that an *hypergraph* $\mathcal{H}$ is a set family $\mathcal{H}\subseteq 2^V$, with *vertex set* V. An hypergraph is said to be r-*uniform* if all its *edges* $S\in\mathcal{H}$ have the same cardinality r. A *proper coloring with* t *colors* of $\mathcal{H}$ is a map $c:V\to[t]$ such that there is no monochromatic edge, that is such that in each edge there are two vertices i and j with $c(i)\neq c(j)$. The smallest number t such that there exists such a proper coloring is called *the chromatic number* of $\mathcal{H}$ and denoted by $\chi(\mathcal{H})$.

In 1986, solving a conjecture of Erdős [4], Alon, Frankl and Lovász [2] found the chromatic number of *Kneser hypergraphs*. The Kneser hypergraph $KG^r\binom{[n]}{k}$ is a r-uniform hypergraph which has the k-subsets of $[n]$ as vertex set and whose edges are formed by the r-uple of disjoint k-subsets of $[n]$. Let n,k,r,t be positive integers such that $n\geq(t-1)(r-1)+rk$. Then $\chi\left(KG^r\binom{[n]}{k}\right)>t$. Combined with a lemma by Erdős giving an explicit proper coloring, it implies that $\chi\left(KG^r\binom{[n]}{k}\right)=\left\lceil\frac{n-(k-1)r}{r-1}\right\rceil$. The proof found by Alon, Frankl and Lovász used tools from algebraic topology.

In 2001, Ziegler gave a combinatorial proof of this theorem [13], which makes no use of homology, simplicial approximation,... He was inspired by a combinatorial proof of the Lovász theorem found by Matoušek [9]. A subset $S\subseteq[n]$ is s-*stable* if any two of its elements are at least "at distance s

apart" on the n-cycle, that is, if $s \leq |i-j| \leq n-s$ for distinct $i,j \in S$. Define then $KG^r\binom{[n]}{k}_{s\text{-stab}}$ as the hypergraph obtained by restricting the vertex set of $KG^r\binom{[n]}{k}$ to the s-stable k-subsets. At the end of his paper, Ziegler made the supposition that the chromatic number of $KG^r\binom{[n]}{k}_{r\text{-stab}}$ is equal to the chromatic number of $KG^r\binom{[n]}{k}$ for any $n \geq kr$. This supposition generalizes both Schrijver's theorem and the Alon-Frankl-Lovász theorem. Alon, Drewnowski and Łucsak make this supposition an explicit conjecture in [1].

Conjecture 1. *Let n, k, r be non-negative integers such that $n \geq rk$. Then*

$$\chi\left(KG^r\binom{[n]}{k}_{r\text{-}stab}\right) = \left\lceil \frac{n-(k-1)r}{r-1} \right\rceil.$$

We prove a weaker form of this statement, but which strengthes the Alon-Frankl-Lovász theorem. Let $V(n,k,s)$ be the set of k-subsets S of $[n]$ such that for all $i,j \in S$, we have $|i-j| \geq s$ We define the almost s-stable Kneser hypergraphs $KG^r\binom{[n]}{k}^{\sim}_{s\text{-stab}}$ to be the r-uniform hypergraph whose vertex set is $V(n,k,s)$ and whose edges are the r-uples of disjoint elements of $V(n,k,s)$.

Theorem 1. *Let p be a prime number and n, k be non negative integers such that $n \geq pk$. We have*

$$\chi\left(KG^p\binom{[n]}{k}^{\sim}_{2\text{-stab}}\right) \geq \left\lceil \frac{n-(k-1)p}{p-1} \right\rceil.$$

Combined with the lemma by Erdős, we get that

$$\chi\left(KG^p\binom{[n]}{k}^{\sim}_{2\text{-stab}}\right) = \left\lceil \frac{n-(k-1)p}{p-1} \right\rceil.$$

Moreover, we will see that it is then possible to derive the following corollary. Denote by $\mu(r)$ the number of prime divisors of r counted with multiplicities. For instance, $\mu(6) = 2$ and $\mu(12) = 3$. We have

Corollary 1. *Let n, k, r be non-negative integers such that $n \geq rk$. We have*

$$KG^r\binom{[n]}{k}^{\sim}_{2^{\mu(r)}\text{-stab}} = \left\lceil \frac{n-(k-1)r}{r-1} \right\rceil.$$

2. Notations and tools

$Z_p = \{\omega, \omega^2, \ldots, \omega^p\}$ is the cyclic group of order p, with generator ω.

We write σ^{n-1} for the $(n-1)$-dimensional simplex with vertex set $[n]$ and by σ^{n-1}_{k-1} the $(k-1)$-skeleton of this simplex, that is the set of faces of σ^{n-1} having k or less vertices.

If A and B are two sets, we write $A \uplus B$ for the set $(A \times \{1\}) \cup (B \times \{2\})$. For two simplicial complexes, K and L, with vertex sets $V(\mathsf{K})$ and $V(\mathsf{L})$, we denote by $\mathsf{K} * \mathsf{L}$ the *join* of these two complexes, which is the simplicial complex having $V(\mathsf{K}) \uplus V(\mathsf{L})$ as vertex set and

$$\{F \uplus G : F \in \mathsf{K}, G \in \mathsf{L}\}$$

as set of faces. We define also K^{*n} to be the join of n disjoint copies of K.

Let $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n$. We denote by $\mathrm{alt}(X)$ the size of the longest alternating subsequence of non-zero terms in X. A sequence $(j_1, j_2, \ldots, j_m)$ of elements of Z_p is said to be *alternating* if any two consecutive terms are different. For instance (assume $p = 5$) $\mathrm{alt}(\omega^2, \omega^3, 0, \omega^3, \omega^5, 0, 0, \omega^2) = 4$ and $\mathrm{alt}(\omega^1, \omega^4, \omega^4, \omega^4, 0, 0, \omega^4) = 2$.

Any element element $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n$ can alternatively and without further mention be denoted by a p-uple $(X_1, \ldots, X_p)$ where $X_j := \{i \in [n] : x_i = \omega^j\}$. Note that the X_j are then necessarily disjoint. For two elements $X, Y \in (Z_p \cup \{0\})^n$, we denote by $X \subseteq Y$ the fact

that for all $j \in [p]$ we have $X_j \subseteq Y_j$. When $X \subseteq Y$, note that the sequence of non-zero terms in $(x_1, \ldots, x_n)$ is a subsequence of $(y_1, \ldots, y_n)$.

The proof of Theorem 1 makes use of a variant of the Z_p-Tucker lemma by Ziegler [13].

Lemma 1 (Z_p-Tucker lemma)**.** *Let p be a prime, $n, m \geq 1$, $\alpha \leq m$ and let*

$$\begin{array}{rccc} \lambda: & (Z_p \cup \{0\})^n \setminus \{(0,\ldots,0)\} & \longrightarrow & Z_p \times [m] \\ & X & \longmapsto & (\lambda_1(X), \lambda_2(X)) \end{array}$$

be a Z_p-equivariant map satisfying the following properties:

- *for all $X^{(1)} \subseteq X^{(2)} \in (Z_p \cup \{0\})^n \setminus \{(0,\ldots,0)\}$, if $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \leq \alpha$, then $\lambda_1(X^{(1)}) = \lambda_1(X^{(2)})$;*
- *for all $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \in (Z_p \cup \{0\})^n \setminus \{(0,\ldots,0)\}$, if $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = \lambda_2(X^{(p)}) \geq \alpha + 1$, then the $\lambda_1(X^{(i)})$ are not pairwise distinct for $i = 1, \ldots, p$.*

Then $\alpha + (m - \alpha)(p - 1) \geq n$.

We can alternatively say that $X \mapsto \lambda(X) = (\lambda_1(X), \lambda_2(X))$ is a Z_p-equivariant simplicial map from $\mathrm{sd}\left(Z_p^{*n}\right)$ to $\left(Z_p^{*\alpha}\right) * \left((\sigma_{p-2}^{p-1})^{*(m-\alpha)}\right)$, where $\mathrm{sd}(\mathsf{K})$ denotes the fist barycentric subdivision of a simplicial complex K.

Proof of the Z_p-Tucker lemma. According to Dold's theorem [3, 8], if such a map λ exists, the dimension of $\left(Z_p^{*\alpha}\right) * \left((\sigma_{p-2}^{p-1})^{*(m-\alpha)}\right)$ is strictly larger than the connectivity of Z_p^{*n}, that is $\alpha + (m - \alpha)(p - 1) - 1 > n - 2$. □

It is also possible to give a purely combinatorial proof of this lemma through the generalized Ky Fan theorem from [5].

3. Proof of the main results

Proof of Theorem 1. We follow the scheme used by Ziegler in [13]. We endow $2^{[n]}$ with an arbitrary linear order $\preceq$.

Assume that $KG^p\binom{[n]}{k}_{2\text{-stab}}^{\sim}$ is properly colored with C colors $\{1, \ldots, C\}$. For $S \in V(n, k, 2)$, we denote by $c(S)$ its color. Let $\alpha = p(k - 1)$ and $m = p(k - 1) + C$.

Let $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. We can write alternatively $X = (X_1, \ldots, X_p)$.

- if $\mathrm{alt}(X) \leq p(k-1)$, let j be the index of the X_j containing the smallest integer (ω^j is then the first non-zero term in $(x_1, \ldots, x_n)$), and define
$$\lambda(X) := (j, \mathrm{alt}(X)).$$
- if $\mathrm{alt}(X) \geq p(k - 1) + 1$: in the longest alternating subsequence of non-zero terms of X, at least one of the elements of Z_p appears at least k times; hence, in at least one of the X_j there is an element S of $V(n, k, 2)$; choose the smallest such S (according to $\preceq$). Let j be such that $S \subseteq X_j$ and define
$$\lambda(X) := (j, c(S) + p(k - 1)).$$

λ is Z_p-equivariant map from $(Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$ to $Z_p \times [m]$.

Let $X^{(1)} \subseteq X^{(2)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. If $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \leq \alpha$, then the longest alternating subsequences of non-zero terms of $X^{(1)}$ and $X^{(2)}$ have same size. Clearly, the first non-zero terms of $X^{(1)}$ and $X^{(2)}$ are equal.

Let $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. If $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = \lambda_2(X^{(p)}) \geq \alpha + 1$, then for each $i \in [p]$ there is $S_i \in V(n, k, 2)$ and $j_i \in [p]$ such that we have $S_i \subseteq X_{j_i}^{(i)}$ and $\lambda_2(X^{(i)}) = c(S_i)$. If all $\lambda_1(X^{(i)})$ would be distinct, then it would mean that all j_i

would be distinct, which implies that the S_i would be disjoint but colored with the same color, which is impossible since c is a proper coloring.

We can thus apply the Z_p-Tucker lemma (Lemma 1) and conclude that $n \leq p(k-1) + C(p-1)$, that is

$$C \geq \left\lceil \frac{n-(k-1)p}{p-1} \right\rceil .$$

$\square$

To prove Corollary 1, we prove the following lemma, both statement and proof of which are inspired by Lemma 3.3 of [1].

Lemma 2. *Let r_1, r_2, s_1, s_2 be non-negative integers ≥ 1, and define $r = r_1 r_2$ and $s = s_1 s_2$.*

Assume that for $i = 1, 2$ we have $\chi\left(KG^{r_i}\binom{[n]}{k}^{\sim}_{s_i\text{-stab}}\right) = \left\lceil \frac{n-(k-1)r_i}{r_i-1} \right\rceil$ for all integers n and k such that $n \geq r_i k$.

Then we have $\chi\left(KG^{r}\binom{[n]}{k}^{\sim}_{s\text{-stab}}\right) = \left\lceil \frac{n-(k-1)r}{r-1} \right\rceil$ for all integers n and k such that $n \geq rk$.

Proof. Let $n \geq (t-1)(r-1) + rk$. We have to prove that $\chi\left(KG^{r}\binom{[n]}{k}^{\sim}_{s\text{-stab}}\right) > t$. For a contradiction, assume that $KG^r\binom{[n]}{k}_{s\text{-stab}}$ is properly colored with $C \leq t$ colors. For $S \in V(n,k,p)$, we denote by $c(S)$ its color. We wish to prove that there are $S_1, \ldots, S_r$ disjoint elements of $V(n,k,s)$ with $c(S_1) = \ldots = c(S_r)$.

Take $A \in V(n, n_1, s_1)$, where $n_1 := r_1 k + (t-1)(r_1 - 1)$. Denote $a_1 < \ldots < a_{n_1}$ the elements of A and define $h : V(n_1, k, s_2) \to [t]$ as follows: let $B \in V(n_1, k, s_2)$; the k-subset $S = \{a_i : i \in B\} \subseteq [n]$ is an element of $V(n,k,s)$, and gets as such a color $c(S)$; define $h(B)$ to be this $c(S)$. Since $n_1 = r_1 k + (t-1)(r_1-1)$, there are $B_1, \ldots, B_{r_1}$ disjoint elements of $V(n_1, k, s_2)$ having the same color by h. Define $\tilde{h}(A)$ to be this common color.

Make the same definition for all $A \in V(n, n_1, s_1)$. The map $\tilde{h}$ is a coloring of $KG^{r_2}\binom{[n]}{n_1}^{\sim}_{s_1\text{-stab}}$ with t colors. Now, note that

$$(t-1)(r-1) + rk = (t-1)(r_1 r_2 - r_2 + r_2 - 1) + r_1 r_2 k = (t-1)(r_2 - 1) + r_2((t-1)(r_1-1) + r_1 k)$$

and thus that $n \geq (t-1)(r_2-1) + r_2 n_1$. Hence, there are $A_1, \ldots, A_{r_2}$ disjoint elements of $V(n, n_1, s_1)$ with the same color. Each of the A_i gets its color from r_1 disjoint elements of $V(n,k,s)$, whence there are $r_1 r_2$ disjoint elements of $V(n,k,s)$ having the same color by the map c. $\square$

Proof of Corollary 1. Direct consequence of Theorem 1 and Lemma 2. $\square$

4. Short combinatorial proof of Schrijver's theorem

Recall that Schrijver's theorem is

Theorem 2. *Let $n \geq 2k$.* $\chi\left(KG\binom{[n]}{k}_{2\text{-stab}}\right) = n - 2k + 2$.

When specialized for $p = 2$, Theorem 1 does not imply Schrijver's theorem since the vertex set is allowed to contain subsets with 1 and n together. Anyway, by a slight modification of the proof, we can get a short combinatorial proof of Schrijver's theorem. Alternative proofs of this kind – but not that short – have been proposed in [10, 13]

For a positive integer n, we write $\{+,-,0\}^n$ for the set of all *signed subsets* of $[n]$, that is, the family of all pairs (X^+, X^-) of disjoint subsets of $[n]$. Indeed, for $X \in \{+,-,0\}^n$, we can define $X^+ := \{i \in [n] : X_i = +\}$ and analogously X^-.

We define $X \subseteq Y$ if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$.

By $\operatorname{alt}(X)$ we denote the length of the longest alternating subsequence of non-zero signs in X. For instance: $\operatorname{alt}(+0--+0-) = 4$, while $\operatorname{alt}(--++-+0+-) = 5$.

The proof makes use of the following well-known lemma see [8, 12, 13] (which is a special case of Lemma 1 for $p = 2$).

Lemma 3 (Tucker's lemma)**.** *Let* $\lambda : \{-, 0, +\}^n \setminus \{(0, 0, \ldots, 0)\} \to \{-1, +1, \ldots, -n, +n\}$ *be a map such that* $\lambda(-X) = -\lambda(X)$. *Then there exist* A, B *in* $\{-, 0, +\}^n$ *such that* $A \subseteq B$ *and* $\lambda(A) = -\lambda(B)$.

Proof of Schrijver's theorem. The inequality $\chi\left(KG^2\binom{[n]}{k}_{\text{2-stab}}\right) \leq n-2k+2$ is easy to prove (with an explicit coloring) and well-known. So, to obtain a combinatorial proof, it is sufficient to prove the reverse inequality.

Let us assume that there is a proper coloring c of $KG^2\binom{[n]}{k}_{\text{2-stab}}$ with $n - 2k + 1$ colors. We define the following map λ on $\{-, 0, +\}^n \setminus \{(0, 0, \ldots, 0)\}$.

- if $\operatorname{alt}(X) \leq 2k - 1$, we define $\lambda(X) = \pm\operatorname{alt}(X)$, where the sign is determined by the first sign of the longest alternating subsequence of X (which is actually the first non zero term of X).
- if $\operatorname{alt}(X) \geq 2k$, then X^+ and X^- both contain a stable subset of $[n]$ of size k. Among all stable subsets of size k included in X^- and X^+, select the one having the smallest color. Call it S. Then define $\lambda(X) = \pm(c(S) + 2k - 1)$ where the sign indicates which of X^- or X^+ the subset S has been taken from. Note that $c(S) \leq n - 2k$.

The fact that for any $X \in \{-, 0, +\}^n \setminus \{(0, 0, \ldots, 0)\}$ we have $\lambda(-X) = -\lambda(X)$ is obvious. λ takes its values in $\{-1, +1, \ldots, -n, +n\}$. Now let us take A and B as in Tucker's lemma, with $A \subseteq B$ and $\lambda(A) = -\lambda(B)$. We cannot have $\operatorname{alt}(A) \leq 2k - 1$ since otherwise we will have a longest alternating in B containg the one of A, of same length but with a different sign. Hence $\operatorname{alt}(A) \geq 2k$. Assume w.l.o.g. that $\lambda(A)$ is defined by a stable subset $S_A \subseteq A^-$. Then the stable subset S_B defining $\lambda(B)$ is such that $S_B \subseteq B^+$, which implies that $S_A \cap S_B = \emptyset$. We have moreover $c(S_A) = |\lambda(A)| = |\lambda(B)| = c(S_B)$, but this contradicts the fact that c is proper coloring of $KG^2\binom{[n]}{k}_{\text{2-stab}}$. □

5. Concluding remarks

We have seen that one of the main ingredients is the notion of alternating sequence of elements in Z_p. Here, our notion only requires that such an alternating sequence must have $x_i \neq x_{i+1}$. To prove Conjecture 1, we need probably something stronger. For example, a sequence is said to be alternating if any p consecutive terms are all distinct. Anyway, all our attempts to get something through this approach have failed.

Recall that Alon, Drewnowski and Łucsak [1] proved Conjecture 1 when r is a power of 2. With the help of a computer and `lpsolve`, we check that Conjecture 1 is moreover true for

- $n \leq 9$, $k = 2$, $r = 3$.
- $n \leq 12$, $k = 3$, $r = 3$.
- $n \leq 14$, $k = 4$, $r = 3$.
- $n \leq 13$, $k = 2$, $r = 5$.
- $n \leq 16$, $k = 3$, $r = 5$.
- $n \leq 21$, $k = 4$, $r = 5$.

References

1. N. Alon, L. Drewnowski, and T Łuczak, *Stable Kneser hypergraphs and ideals in* $\mathbb{N}$ *with the Nikodým property*, Proceedings of the American mathematical society **137** (2009), 467–471.
2. N. Alon, P. Frankl, and L. Lovász, *The chromatic number of Kneser hypergraphs*, Transactions Amer. Math. Soc. **298** (1986), 359–370.
3. A. Dold, *Simple proofs of some Borsuk-Ulam results*, Contemp. Math. **19** (1983), 65–69.

4. P. Erdős, *Problems and results in combinatorial analysis*, Colloquio Internazionale sulle Teorie Combinatorie (Rome 1973), Vol. II, No. 17 in Atti dei Convegni Lincei, 1976, pp. 3–17.
5. B. Hanke, R. Sanyal, C. Schultz, and G. Ziegler, *Combinatorial stokes formulas via minimal resolutions*, Journal of Combinatorial Theory, series A (to appear).
6. M. Kneser, *Aufgabe 360*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 2. Abteilung, vol. 50, 1955, p. 27.
7. L. Lovász, *Knesers conjecture, chromatic number and homotopy*, Journal of Combinatorial Theory, Series A **25** (1978), 319–324.
8. J. Matoušek, *Using the Borsuk-Ulam theorem*, Springer Verlag, Berlin–Heidelberg–New York, 2003.
9. ______, *A combinatorial proof of Kneser's conjecture*, Combinatorica **24** (2004), 163–170.
10. F. Meunier, *Combinatorial Stokes formulae*, European Journal of Combinatorics **29** (2008), 286–297.
11. A. Schrijver, *Vertex-critical subgraphs of Kneser graphs*, Nieuw Arch. Wiskd., III. Ser. **26** (1978), 454–461.
12. A. W. Tucker, *Some topological properties of disk and sphere*, Proceedings of the First Canadian Mathematical Congress, Montreal 1945, 1946, pp. 285–309.
13. G. Ziegler, *Generalized Kneser coloring theorems with combinatorial proofs*, Invent. Math. **147** (2002), 671–691.

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